All of Chapter 2 except page 74 Chapter Exercise B

P35

[1] $\overrightarrow{BC} = \overrightarrow{AD}$, $\overrightarrow{BA} = \overrightarrow{CD}$

P36

[2] EQUAL: $\vec{d} = \vec{f}$. INVERSE: $\vec{b}$, $\vec{e}$

Revisions

✓ P. 55 #5

P. 38 #6

P. 39 #8

P. 63 missing

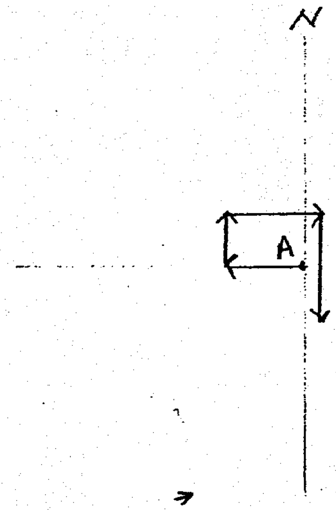
✓ P. 67 #1 better proof not using thmd geometry

p74 Exercise B missing

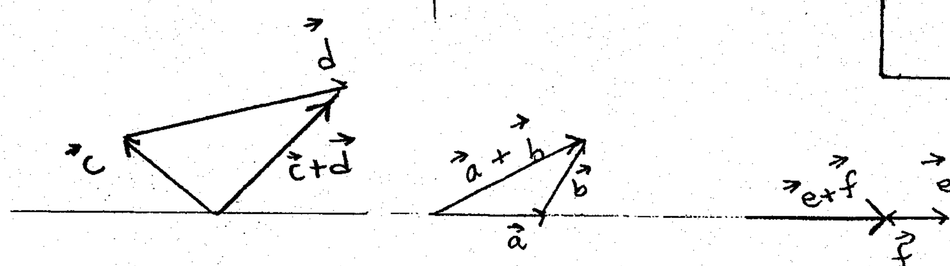
✓ p 67 #2

P37

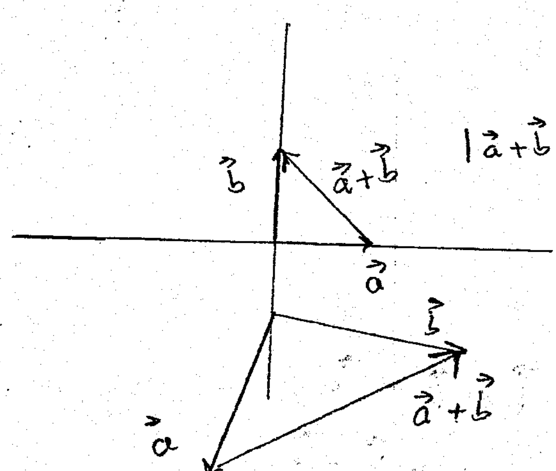
[1]



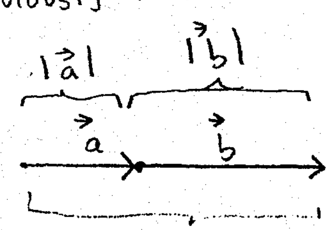
[2]



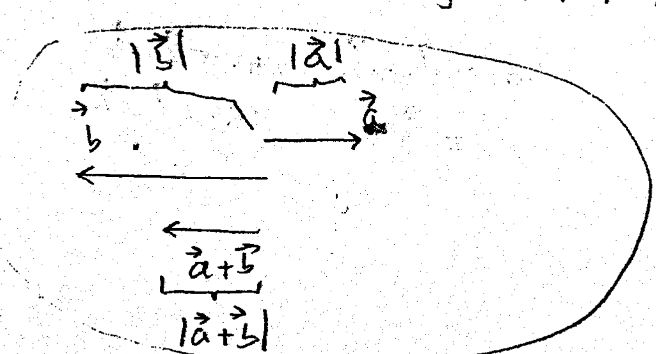
[3]



$|\vec{a}+\vec{b}|$ obviously less than $|\vec{a}| + |\vec{b}|$.



$|\vec{a}+\vec{b}|$ equal $|\vec{a}| + |\vec{b}|$



Hard to imagine how $|\vec{a}| + |\vec{b}|$ could be less than $|\vec{a}+\vec{b}|$. It can not.

P38

[4] $\vec{a}$ followed by $\vec{b}$ equivalent to $\vec{c}$

$\vec{b}$ " " $\vec{a}$ " " $\vec{c}$

$\vec{a}$ followed by ($\vec{b}$ followed by $\vec{c}$) is $\overrightarrow{AD}$

($\vec{a}$ followed by $\vec{b}$) followed by $\vec{c}$ is $\overrightarrow{AD}$

[5] Prove $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

Proof

$$
\begin{aligned}
LHS &= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} \\
&= \overrightarrow{AC} + \overrightarrow{CA} \\
&= \overrightarrow{AC} - \overrightarrow{AC} \\
&= \vec{0}
\end{aligned}
$$

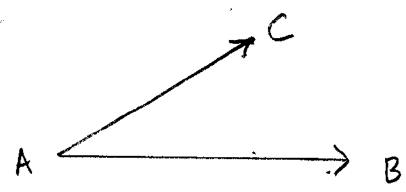
P 39

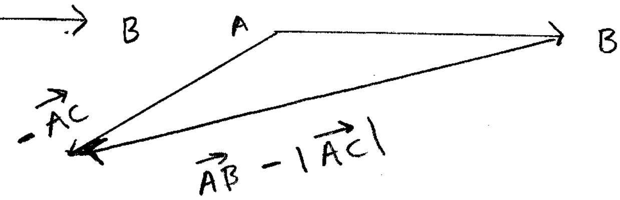
[6] Let $\vec{a} - \vec{b} = \vec{x}$. check $\vec{x}$ such that $\vec{b} + \vec{x} = \vec{a}$.

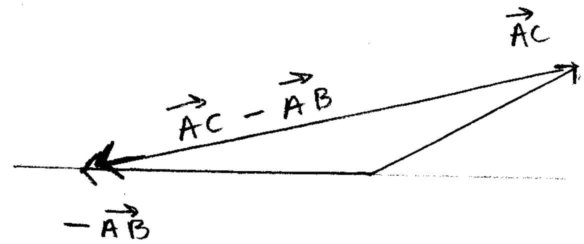
$$
\begin{aligned}
LHS &= \vec{a} - \vec{b} \\
&= \vec{a} + (-\vec{b}) && \text{since} \\
&= \vec{b} + \vec{x} - \vec{b} && \text{assoc and comm.} \\
&= \vec{x} \\
&= RHS
\end{aligned}
$$

P39, ctd

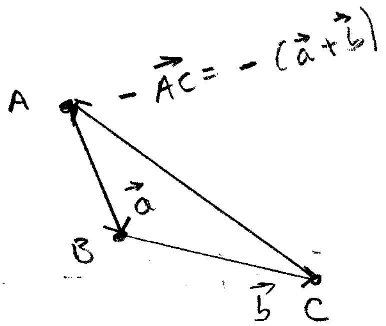
[7]



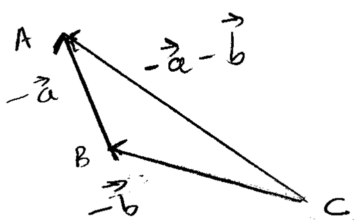




[8]



$$\left.\begin{array}{l}\vec{a} = \vec{AB} \\ \vec{b} = \vec{BC}\end{array}\right\} \text{then} \quad -(\vec{a}+\vec{b}) = -\vec{AC}$$

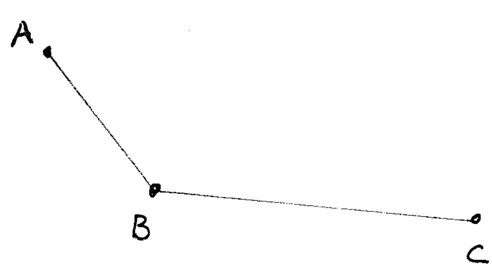


Trip from C to B to A is

$-\vec{BC} - \vec{AB} = -\vec{AC}$

$\vec{CB} + \vec{BA} = \vec{CA}$

Think,



$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$-(\vec{AB} + \vec{BC}) = -\vec{AC} = \vec{CA}$$

and

$$\begin{aligned} -\vec{AB} - \vec{BC} &= \vec{BA} + \vec{CB} \\ &= \vec{CB} + \vec{BA} \\ &= \vec{CA} \\ &= -\vec{AC} \end{aligned}$$

P41

[1] $\vec{AB} = m\vec{AC}$ then A, B, C co-linear.

Proof: Since $\vec{AB} = m\vec{AC}$, $\vec{AB} \parallel \vec{AC}$. A and B lie on line l_1 and A and C lie on line l_2. Only one line can be drawn through a given point parallel to a given line, so l_1 and l_2 are the same line. therefore, A, B, C co-linear. $\square$

P42

[2] Checked.

[3.1] $3\vec{a} + 4\vec{a} - \vec{a} = (3+4-1)\vec{a} = 6\vec{a}$

[3.2] $3(\vec{u} + 2\vec{v}) - 3(\vec{u} - 4\vec{v})$

$= 3\vec{u} + 6\vec{v} - 3\vec{u} + 12\vec{v}$

$= 18\vec{v}$

[4] Let $\vec{a} = \vec{AB}$, $b = \vec{AC}$

$\vec{NL} = \frac{1}{2}\vec{b}$

$\vec{LM} = \frac{1}{2}(-\vec{a}) = -\frac{1}{2}\vec{a}$

$\vec{MN} = \frac{1}{2}\vec{a} - \frac{1}{2}\vec{b} = \frac{1}{2}(\vec{a} - \vec{b})$

$\vec{AL} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b} = \frac{1}{2}(\vec{a} + \vec{b})$

$\vec{BM} = -\vec{a} + \frac{1}{2}\vec{b}$

$\vec{CN} = -b + \frac{1}{2}\vec{a}$

p42, ctd

[5] check $\vec{e} = \frac{\vec{a}}{|\vec{a}|}$, $\vec{a} \neq \vec{0}$.

Prove that $\vec{e} = \frac{\vec{a}}{|\vec{a}|}$ is a unit vector with same direction as $\vec{a}$.

Proof

$\frac{1}{|\vec{a}|}$ is a scalar, call it m. So,

$\frac{\vec{a}}{|\vec{a}|} = m\vec{a}$. Thus, $\frac{\vec{a}}{|\vec{a}|} \parallel \vec{a}$. I.e. $\boxed{\vec{e} \parallel \vec{a}.}$

Now $|\vec{e}| = \left|\frac{\vec{a}}{|\vec{a}|}\right| = \frac{|\vec{a}|}{||\vec{a}||} = \frac{|\vec{a}|}{|\vec{a}|} = 1.$

Therefore $\vec{e} = \frac{\vec{a}}{|\vec{a}|}$ is a unit vector in the same direction as $\vec{a}$.

$\square$

P44

[1] $A(-2,-6)$, $B(3,1)$, $C(3,4)$

$$\overrightarrow{AB} = 5\hat{e}_1 + 7\hat{e}_2$$

$$\overrightarrow{BC} = 3\hat{e}_2$$

$$\overrightarrow{CA} = -5\hat{e}_1 - 10\hat{e}_2$$

P45

[2]

$$\vec{a} = -3\hat{e}_2$$

$$\vec{b} = -2\hat{e}_1$$

$$\vec{c} = 4\hat{e}_1 + 2\hat{e}_2$$

$$\vec{d} = 3\hat{e}_1 - 3\hat{e}_2$$

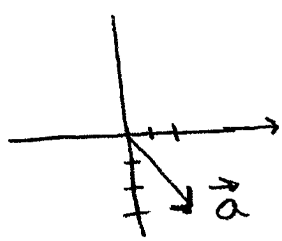
[3]

$$\overrightarrow{AB} = \langle 5, 3 \rangle$$

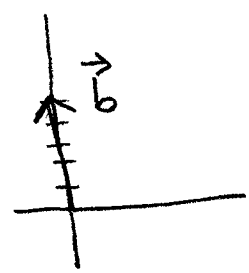
$$\overrightarrow{BA} = \langle -5, -3 \rangle$$

P46

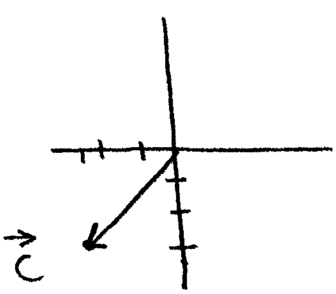
[4.1]



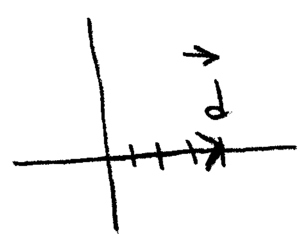
[4.2]



[4.3]



[4.4]



[5] a) Derive: $\langle a_1, a_2\rangle - \langle b_1, b_2\rangle = \langle a_1 - b_1, a_2 - b_2\rangle$

$$\vec{a} = a_1\hat{e}_1 + a_2\hat{e}_2$$

$$\vec{b} = b_1\hat{e}_1 + b_2\hat{e}_2$$

$$\vec{a} - \vec{b} = (a_1 - b_1)\hat{e}_1 + (a_2 - b_2)\hat{e}_2$$

$$= \langle a_1 - b_1, a_2 - b_2\rangle$$

b) derive $m\langle a_1, a_2\rangle = \langle ma_1, ma_2\rangle$

$$\vec{a} = a_1\hat{e}_1 + a_2\hat{e}_2$$

$$m\vec{a} = ma_1\hat{e}_1 + ma_2\hat{e}_2$$

$$= \langle ma_1, ma_2\rangle$$

[6.0] $\vec{a} = \langle 2, -3\rangle$ $\vec{b} = \langle -1, 2\rangle$ $\vec{c} = \langle 5, 0\rangle$

[6.1] $\vec{u} = \vec{a} + \vec{b} + \vec{c} = \langle 2, -3\rangle + \langle -1, 2\rangle + \langle 5, 0\rangle$

$$= \langle 2 - 1 + 5, -3 + 2 + 0\rangle$$

$$= \langle 6, -1\rangle$$

[6.2] $\vec{v} = -3\vec{a} + 2\vec{b} + \vec{c}$

$$= -3\langle 2, -3\rangle + 2\langle -1, 2\rangle + \langle 5, 0\rangle$$

$$= \langle -6, 9\rangle + \langle -2, 4\rangle + \langle 5, 0\rangle$$

$$= \langle -6 - 2 + 5, 9 + 4 + 0\rangle$$

$$= \langle -3, 13\rangle$$

$\rightarrow$

P47, ctd

[6.3] $\vec{w} = 3(\vec{a}+\vec{b}) - 2(\vec{b}-\vec{c})$

$= 3\langle 1,-1\rangle - 2\langle -6, 2\rangle$

$= \langle 3, -3\rangle + \langle 12 - 4\rangle$

$= \langle 3+12, -3-4\rangle$

$= \langle 15, -7\rangle$

[7] $\vec{a} = \langle 2,-3\rangle, \quad \vec{b} = \langle -1, 2\rangle, \quad c = \langle 5,0\rangle$

$|\vec{a}| = \sqrt{4+9} = \sqrt{13}$

$|\vec{b}| = \sqrt{1+4} = \sqrt{5}$

$|\vec{c}| = \sqrt{25+0} = 5$

P 49

[1] SHOW $\theta = 0°$, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos 0°$

$= |\vec{a}||\vec{b}|$

Show $\theta = 180°$, $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos 180$

$= |\vec{a}||\vec{b}|(-1)$

$= -|\vec{a}||\vec{b}|$

[2.1] $\overrightarrow{AB} \cdot \overrightarrow{AC} = 2\sqrt{3} \cos 30 = 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 3$

[2.2] $\overrightarrow{CA} \cdot \overrightarrow{CB} = 0$, $\because \angle C = 90^\circ$ and $\cos 90 = 0$

[2.3] $\overrightarrow{AB} \cdot \overrightarrow{BC} = -\overrightarrow{BA} \cdot \overrightarrow{BC} = -(2)(1)\cos 60 = -2\left(\frac{1}{2}\right) = -1$

[2.4] $\overrightarrow{AB} \cdot \overrightarrow{CA} = \overrightarrow{AB} \cdot (-\overrightarrow{AC}) = -2\sqrt{3}\cos 30 = -2\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = -3$

[3.1] $\overrightarrow{AB} \cdot \overrightarrow{CB}$

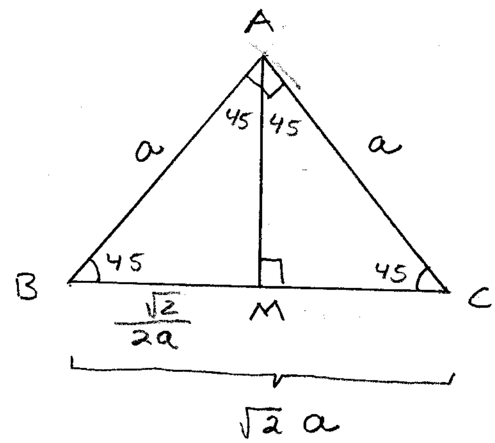
$= -\overrightarrow{BA} \cdot (-\overrightarrow{BC})$

$= \overrightarrow{BA} \cdot \overrightarrow{BC}$

$= (a)(\sqrt{2}\,a) \cos 45$

$= a^2\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)$

$= a^2$



[3.2] $\overrightarrow{BA} \cdot \overrightarrow{CA} = -\overrightarrow{AB} \cdot (-\overrightarrow{AC}) = \overrightarrow{AB} \cdot \overrightarrow{AC} = 0$, $\because \angle A = 90^\circ$

[3.3] $\overrightarrow{AM} \cdot \overrightarrow{BA} = \overrightarrow{AM} \cdot (-\overrightarrow{AB}) = -\overrightarrow{AM} \cdot \overrightarrow{AB}$

$= -\left(\frac{\sqrt{2}}{2a}\right)(a) \cos 45^\circ$

$= -\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}$

$= -\frac{1}{2}$

P 52

[4.1] $\vec{a} \cdot \vec{b} = (-2)(5) + (3)(4) = -10 + 12 = 2$

[4.2] $\vec{a} \cdot \vec{b} = (3)(-4) + (5)(1) = -12 + 5 = -7$

[5.0] $\vec{a} = \langle 3, -4 \rangle, \vec{b} = \langle 8, 6 \rangle$ Show $\vec{a} \perp \vec{b}$

$\vec{a} \cdot \vec{b} = (3)(8) + (-4)(6) = 0, \quad \therefore \vec{a} \perp \vec{b}$

[6.1] $$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{(-3)(-1) + (0)(\sqrt{3})}{(3)(2)} = \frac{3}{6} = \frac{1}{2}$$

$$\cos^{-1}\left(\tfrac{1}{2}\right) \Rightarrow \boxed{\theta = 60^\circ}$$

[6.2] $$\cos\theta = \frac{(2)(3) + (1)(-6)}{\sqrt{5} \cdot \sqrt{45}} = 0$$

$$\therefore \theta = 90^\circ$$

P 53

[7.1] Prove: $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

<u>Proof</u>

$$\begin{aligned} \text{LHS} &= (\vec{a} + \vec{b}) \cdot \vec{c} \\ &= \langle a_1 + b_1, a_2 + b_2 \rangle \cdot \langle c_1, c_2 \rangle \\ &= (a_1 + b_1) c_1 + (a_2 + b_2) c_2 \\ &= a_1 c_1 + b_1 c_1 + a_2 c_2 + b_2 c_2 \\ &= a_1 c_1 + a_2 c_2 + b_1 c_1 + b_2 c_2 \\ &= \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} \end{aligned}$$

$\square$

p 53, ctd

[7.2] Prove: $\vec{a} \cdot m\vec{b} = m(\vec{a} \cdot \vec{b})$

Proof

$$
\begin{aligned}
LHS &= \vec{a} \cdot m\vec{b} \\
&= \langle a_1, a_2 \rangle \cdot m \langle b_1, b_2 \rangle \\
&= \langle a_1, a_2 \rangle \cdot \langle mb_1, mb_2 \rangle \\
&= a_1 m b_1 + a_2 m b_2 \\
&= m(a_1 b_1 + a_2 b_2) \\
&= m(\vec{a} \cdot \vec{b}) \\
&= RHS
\end{aligned}
$$

□

[7.3] Prove: $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b})$

Proof

$$
\begin{aligned}
LHS &= (m\vec{a}) \cdot \vec{b} \\
&= \langle m.a_1, ma_2 \rangle \cdot \langle b_1, b_2 \rangle \\
&= ma_1 b_1 + ma_2 b_2 \\
&= m(a_1 b_1 + a_2 b_2) \\
&= m(\vec{a} \cdot \vec{b}) \\
&= RHS
\end{aligned}
$$

□

[8.1] Prove $\vec{a} \cdot (\vec{b} - \vec{c}) = \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}$

proof

{NOTE: We just proved $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$, which says dot product is Right distributive, we'll prove dot product is left distributive for $(\vec{a} + \vec{b})$, then prove the immediate thm by merely substituting $-\vec{b}$ for $\vec{b}$.}

Prove $\vec{c} \cdot (\vec{a} + \vec{b}) = \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b}$

Proof

$$
\begin{aligned}
LHS &= \langle c_1, c_2 \rangle \cdot \langle a_1 + b_1, a_2 + b_2 \rangle \\
&= c_1(a_1 + b_1) + c_2(a_2 + b_2) \\
&= c_1 a_1 + c_1 b_1 + c_2 a_2 + c_2 b_2 \\
&= c_1 a_1 + c_2 a_2 + c_1 b_1 + c_2 b_2 \\
&= \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} \\
&= RHS
\end{aligned}
$$

$\square$

Now,

$$
\begin{aligned}
\vec{a} \cdot (\vec{b} - \vec{c}) &= \vec{a} \cdot (\vec{b} + (-\vec{c})) \\
&= \vec{a} \cdot \vec{b} + \vec{a} \cdot (-\vec{c}) \\
&= \vec{a} \cdot \vec{b} + \vec{a} \cdot (-1)\vec{c} \\
&= \vec{a} \cdot \vec{b} + (-1)\, \vec{a} \cdot \vec{c} \\
&= \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}
\end{aligned}
$$

$\square$

p 53, ctd

[8, 2] Prove: $(\vec{a} - \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{c}$

Proof

$$\begin{aligned} LHS &= (\vec{a} - \vec{b}) \cdot \vec{c} \\ &= (\vec{a} + (-\vec{b})) \cdot \vec{c} \\ &= \vec{a} \cdot \vec{c} + (-\vec{b}) \cdot \vec{c} \\ &= \vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{c} \end{aligned}$$

$\square$

p 54

[9] Let: $|\vec{a}| = 2$, $|\vec{b}| = 3$, $\vec{a} \cdot \vec{b} = 4$

Find $|2\vec{a} - 3\vec{b}|$

SOLN

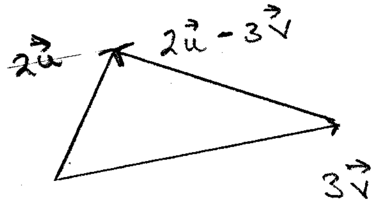
$$\begin{aligned} |2\vec{a} - 3\vec{b}|^2 &= \langle 2\vec{a} - 3\vec{b} \rangle \cdot \langle 2\vec{a} - 3\vec{b} \rangle \\ &= 2\vec{a} \cdot \langle 2\vec{a} - 3\vec{b} \rangle - 3\vec{b} \cdot \langle 2\vec{a} - 3\vec{b} \rangle \\ &= 2\vec{a} \cdot 2\vec{a} - 2\vec{a} \cdot 3\vec{b} - 2a \cdot 3\vec{b} + 3\vec{b} \cdot 3\vec{b} \\ &= 4|\vec{a}|^2 - 12\,\vec{a} \cdot \vec{b} + 9|\vec{b}|^2 \\ &= 4(4) - 12(4) + 9(9) \\ &= 16 - 48 + 81 \\ &= 49 \end{aligned}$$

$\therefore |2\vec{a} - 3\vec{b}| = \sqrt{49} = 7$

P 55

[1.1] $6\vec{u} - 5\vec{v} - 4\vec{u} + 2\vec{v}$

$= 2\vec{u} - 3\vec{v}$

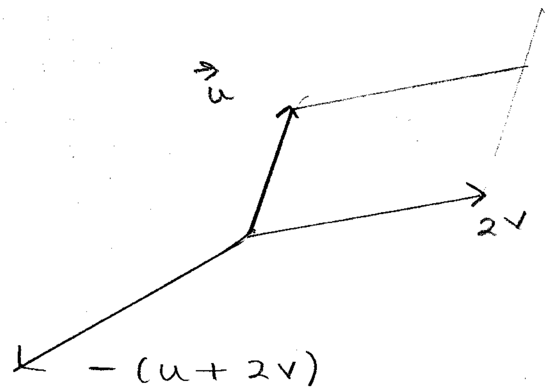


[1.2] $7(\vec{u} - 2\vec{v}) - 4(2\vec{u} - 3\vec{v})$

$= 7\vec{u} - 14\vec{v} - 8u + 12V$

$= -u - 2V$

$= -(u + 2v)$



[2] $\vec{c} = k\vec{a} + l\vec{b}$, $\vec{a} = \langle -2, 3 \rangle$, $\vec{b} = \langle 1, -4 \rangle$, $\vec{c} = \langle 8, -17 \rangle$

<u>SOLN</u>

$k\vec{a} + l\vec{b} = \vec{c}$

$\langle ka_1, ka_2 \rangle + \langle lb_1, lb_2 \rangle = \langle 8, -17 \rangle$

$\Rightarrow \begin{bmatrix} ka_1 + lb_1 = 8 \\ ka_2 + lb_2 = -17 \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} -2k + l = 8 \\ 3k - 4l = -17 \end{bmatrix} \Rightarrow \begin{bmatrix} -8k + 4l = 32 \\ 3k - 4l = -17 \end{bmatrix}$

$\Rightarrow -5k = 15 \Rightarrow \boxed{k = -3}$

$-2(-3) + l = 8$

$6 + l = 8$

$\boxed{l = 2}$

P 55, ctd

[3.1] Prove: $(4\vec{a} + 3\vec{b}) \bullet (4\vec{a} - 3\vec{b}) = 16|\vec{a}|^2 - 9|\vec{b}|^2$

Proof

$$\begin{aligned}
LHS &= (4\vec{a} + 3\vec{b}) \bullet (4\vec{a} - 3\vec{b}) \\
&= 16\,\vec{a}\bullet\vec{a} - 9\,\vec{b}\circ\vec{b} \\
&= 16\,|\vec{a}|^2 - 9\,|\vec{b}|^2 \\
&= RHS
\end{aligned}$$

□

[3.2] Prove: $|\vec{a}+\vec{b}|^2 - |\vec{a}-\vec{b}|^2 = 4\vec{a}\bullet\vec{b}$

Proof

$$\begin{aligned}
LHS &= |\vec{a}+\vec{b}|^2 - |\vec{a}-\vec{b}|^2 \\
&= \langle\vec{a}+\vec{b}\rangle \bullet \langle\vec{a}+\vec{b}\rangle - \langle\vec{a}-\vec{b}\rangle \bullet \langle\vec{a}-\vec{b}\rangle \\
&= \cancel{\vec{a}\cdot\vec{a}} + 2\vec{a}\cdot\vec{b} + \cancel{\vec{b}\cdot\vec{b}} - \cancel{\vec{a}\cdot\vec{a}} + 2\vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{b} \\
&= 4\,\vec{a}\bullet\vec{b} + \vec{b}\cdot\vec{b} \\
&= RHS
\end{aligned}$$

□

[4.1] $|\vec{a}| = 3,\ |\vec{b}| = 4,\ \vec{a}\bullet\vec{b} = 6$

$$\vec{a}\bullet\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\cos\theta = \frac{6}{12} = \frac{1}{2}$$

$$\therefore\ \theta = 60^\circ$$

P55, ctd

[4.2] $\cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = 45°$

[5] $\vec{u} = 4x\langle 2,1\rangle + \langle -1,2\rangle$

$= \langle 8x, 4x\rangle + \langle -1,2\rangle$

$= \langle 8x-1, 4x+2\rangle$

$\vec{v} = x\langle 2,1\rangle - 3\langle -1,2\rangle$

$= \langle 2x, x\rangle + \langle 3,-6\rangle$

$= \langle 2x+3, x-6\rangle$

$\vec{u}\cdot\vec{v} = \langle 8x-1, 4x+2\rangle \bullet \langle 2x+3, x-6\rangle$

$= (8x-1)(2x+3) + (4x+2)(x-6)$

$= (16x^2+22x-3) + (4x^2-22x-12)$

$= 20x^2 - 15$

$= 5(4x^2-3)$

$\vec{u} \perp \vec{v}$ iff $\vec{u}\cdot\vec{v} = 0$

$5(4x^2-3) = 0$

$\Rightarrow \quad 4x^2 - 3 = 0$

$\Rightarrow \quad x = \pm\frac{\sqrt{3}}{2}$

$\square$

Checked answer. BUT THE FOLLOWING SOLUTION IS BETTER.

$(4x\vec{a}+\vec{b})\cdot(x\vec{a}-3\vec{b}) = 0$

$4x^2|\vec{a}|^2 + x\vec{a}\cdot\vec{b} - 12x\,\vec{a}\cdot\vec{b} - 3|\vec{b}|^2 = 0$

$20x^2 - 15 = 0$ } because $\langle 2,1\rangle\cdot\langle -1,2\rangle = -2+2 = 0$

$\therefore \quad x = \pm\frac{\sqrt{3}}{2}$

P55, ctd

[6] Prove for $\vec{a} \neq 0, \vec{b} \neq 0$, $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| \Rightarrow \vec{a} \perp \vec{b}$

Proof

suppose $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

$$|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$$

$$(\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b}) = (\vec{a}-\vec{b}) \cdot (\vec{a}-\vec{b})$$

$$\vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = \vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$$

$$4\vec{a} \cdot \vec{b} = 0$$

$$\vec{a} \cdot \vec{b} = 0$$

$$\therefore \vec{a} \perp \vec{b}$$

P 56

[7] Show $|\vec{OA} \cdot \vec{e}| = OB$

Proof

call angle AOB angle θ.

$$|\vec{OA} \cdot \vec{e}| = \big| |\vec{OA}||\vec{e}| \cos\theta \big|$$

$$= |\vec{OA}| \, |\cos\theta|$$

$$= |\vec{OA}| \cos\theta \qquad , \cos\theta > 0 \because 0 < \theta < 90^\circ$$

$$= |\vec{OA}| \frac{|\vec{OB}|}{|\vec{OA}|}$$

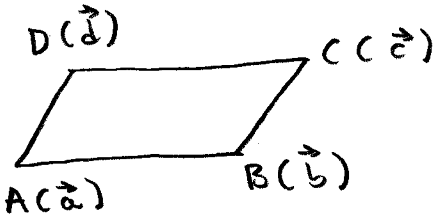
$$= |\vec{OB}|$$

$\square$

JII AG
PPS 57-72
VECTOR APPS

P58

[1]



ABCD parallelogram iff $\begin{cases} \overrightarrow{AB} \parallel \overrightarrow{DC} \\ \text{AND} \\ \overrightarrow{AD} \parallel \overrightarrow{BC} \end{cases}$

$\overrightarrow{AB} \parallel \overrightarrow{DC}$ iff $\overrightarrow{AB} = m\,\overrightarrow{DC}$

$\overrightarrow{AD} \parallel \overrightarrow{BC}$ iff $\overrightarrow{AD} = n\,\overrightarrow{BC}$, $m, n \in \mathbb{R}$

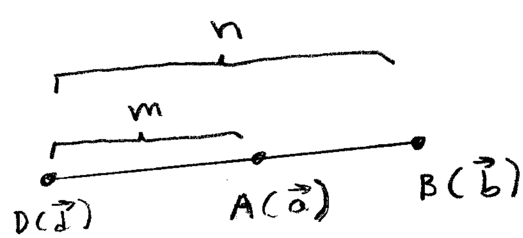
<u>conditions</u> $\quad \vec{b} - \vec{a} = m(\vec{c} - \vec{d})$

and

$\vec{d} - \vec{a} = n(\vec{c} - \vec{b})$, $m, n \in \mathbb{R}$

---

[2]



$$\frac{\overrightarrow{DA}}{\overrightarrow{DB}} = \frac{m}{n}$$

$$\overrightarrow{DA} = \frac{m}{n}\overrightarrow{DB}$$

$$\vec{a} - \vec{d} = \frac{m}{n}(\vec{b} - \vec{d})$$

$$n(\vec{a} - \vec{d}) = m(\vec{b} - \vec{d})$$

$$n\vec{a} - n\vec{d} = m\vec{b} - m\vec{d}$$

$$m\vec{d} - n\vec{d} = m\vec{b} - n\vec{a}$$

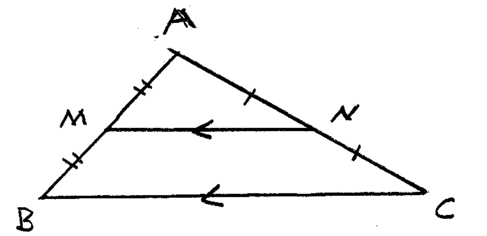
$$\therefore \quad \vec{d} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

$\square$

p 59

[3]

If MN connects midpts of AB and BC, then $MN \parallel BC$ and len $MN = \frac{1}{2}$ len BC.



len $MN = \frac{1}{2}$ len BC

Proof Let $\vec{a}, \vec{b}, \vec{c}, \vec{m}, \vec{n}$ be position vectors of points A, B, C, M, N in ΔABC.

$$\vec{m} = \frac{\vec{a} - \vec{b}}{2}, \quad \vec{n} = \frac{\vec{a} - \vec{c}}{2}$$

$$\begin{aligned}
\overrightarrow{MN} &= \vec{n} - \vec{m} \\
&= \frac{\vec{a} - \vec{c}}{2} - \frac{\vec{a} - \vec{b}}{2} \\
&= \tfrac{1}{2}\left[\vec{a} - \vec{c} - \vec{a} + \vec{b}\right] \\
&= \tfrac{1}{2}\left[-\vec{c} + \vec{b}\right] \\
&= -\tfrac{1}{2}\left[\vec{b} - \vec{c}\right] \\
&= -\tfrac{1}{2}\overrightarrow{CB}
\end{aligned}$$

$$\overrightarrow{MN} = \tfrac{1}{2}\overrightarrow{BC}$$

$\therefore$ MN is $\frac{1}{2}$ len of BC and $MN \parallel BC$.

P60

[4] Thm: Suppose $C(x,y)$ internally divides AB in ratio $m:n$, where $A(x_1,y_1)$, $B(x_2,y_2)$, then

$$x = \frac{mx_2 + nx_1}{m+n}, \quad y = \frac{my_2+ny_1}{m+n}.$$

<u>Proof</u>

$$\vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n},$$ Dem 1 p 58, $\vec{a}, \vec{b}, \vec{c}$ position vectors of A, B, C.

$$\Leftrightarrow \quad \langle x, y\rangle = \frac{m\langle x_2, y_2\rangle + n\langle x_1, y_1\rangle}{m+n}$$

$$= \frac{1}{m+n}\left[\langle mx_2, my_2\rangle + \langle nx_1, ny_1\rangle\right]$$

$$= \frac{1}{m+n}\left[\langle mx_2 + nx_1, my_2 + ny_1\right]$$

so

$$x = \frac{mx_2 + nx_1}{m+n}, \quad y = \frac{my_2 + ny_1}{m+n}$$

$\square$

P61

[1.1] ℓ through $(2,-3)$ with direction vector $\langle 1, 2\rangle$.

$$x = 2 + t$$
$$y = -3 + 2t$$

[1.2] ℓ through $(4,0)$, direction vector $(-3, 2)$

$$x = 4 - 3t$$
$$y = 2t$$

P62

[2.1] $A(-3, 2)$, $B(4,5)$

$$\vec{AB} = \langle 4+3, 5-2\rangle$$
$$= \langle 7, 3\rangle$$

$$\vec{p} = \langle -3, 2\rangle + t\langle 7, 3\rangle$$

[2.2] $A(4,0)$, $B(0,3)$

$$\vec{AB} = \langle 0+4, 3-0\rangle$$
$$= \langle 4, 3\rangle$$

$$\vec{p} = \langle 4, 0\rangle + t\langle 4, 3\rangle$$

[2.3] $A(4,-3)$, $B(-2,5)$

$$\vec{AB} = \langle -2-4, 5+3\rangle$$

$$\vec{p} = \langle 4, -3\rangle + t\langle -6, 8\rangle$$

[2.4] $A(-\frac{2}{3}, -4)$, $B(2,-4)$ $\Rightarrow$ $\vec{AB} = \langle 2+\frac{2}{3}, -4+4\rangle$

$$= \langle \tfrac{8}{3}, 0\rangle$$

$$\vec{p} = \langle -\tfrac{2}{3}, -4\rangle + t\langle \tfrac{8}{3}, 0\rangle$$

P65

[4] $\vec{n} = k\langle 3, -4\rangle$, $k \in \mathbb{R}$, $k \neq 0$.

[5] ℓ through $(5,-4)$ $\perp$ $\vec{n} = \langle 2, 3\rangle$

$$\ell: \; 2(x-5) + 3(y+4) = 0$$

$$2x - 10 + 3y + 12 = 0$$

$$\boxed{2x + 3y + 2 = 0}$$

P66

[6] $d(P \text{ to } \ell)$, $P(2,-5)$, $\ell: 4x - 3y + 7 = 0$

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$$

$$= \frac{4(2) + (3)(5) + 7}{\sqrt{16+9}}$$

$$\boxed{d = 6}$$

$\square$

P67.

[1] Choose center of circle as origin of coordinate system.
Note that since A, B ends of diameter $\vec{a} = -\vec{b}$ and $|\vec{a}| = |\vec{b}| = |\vec{p}|$.

$$\langle \vec{p} - \vec{a}\rangle \cdot \langle \vec{p} - \vec{b}\rangle = \vec{p}\cdot\vec{p} - \vec{a}\cdot\vec{p} - \vec{b}\cdot\vec{p} + \vec{a}\cdot\vec{b}$$

$$= |\vec{p}|^2 - \vec{a}\cdot\vec{p} - \vec{b}\cdot\vec{p} - |\vec{a}|\,|\vec{b}| \quad \Big\}$$ because $\vec{a}, \vec{b}$ ends of diameter $\theta = \pi$, $\cos\pi = -1$

$$= -\vec{a}\cdot\vec{p} - \vec{b}\cdot\vec{p}$$

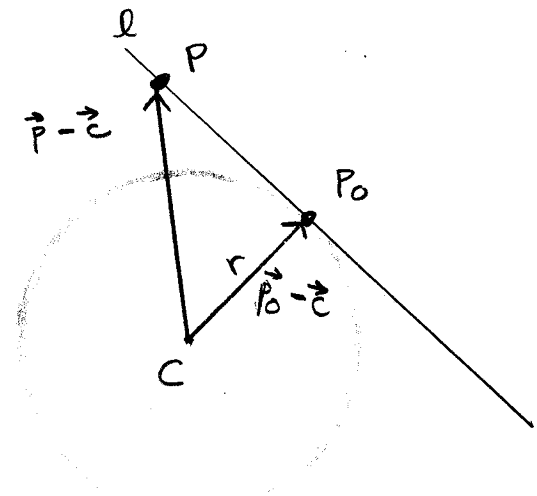
$$= -\vec{p}\cdot(\vec{a} + \vec{b}) \quad \Big\}$$ because $\vec{a}, \vec{b}$ ends of diameter.

$$= -\vec{p}\cdot\vec{0}$$

$$= 0$$

$\square$

P67 **solution 1 (trig**



[2] Prove : P_0 on circle of radius r with line ℓ tangent at P_0, show that vector eqn of ℓ is

$$\langle \vec{p}-\vec{c}\rangle \bullet \langle \vec{P_0}-\vec{c}\rangle = r^2$$

where p is movable point on ℓ.

Proof

$$\begin{aligned}
LHS &= \langle \vec{p}-\vec{c}\rangle \bullet \langle \vec{P_0}-\vec{c}\rangle \\
&= |\vec{p}-\vec{c}|\,|\vec{P_0}-\vec{c}| \cos\theta \\
&= |\vec{p}-\vec{c}|\, r \cos\theta \\
&= |\vec{p}-\vec{c}|\, r \frac{|\vec{P_0}-\vec{c}|}{|\vec{p}-\vec{c}|} \\
&= |\vec{P_0}-\vec{c}|\, r \\
&= r^2 \\
&= R.H.S
\end{aligned}$$

$\square$

Proof without using trigonometry is on following page

P67 Solution 2 (no trig)

[2]

$\ell: \ n \cdot (\vec{p} \cdot \vec{p}_0) = 0$

$\vec{n} = (\vec{c} - \vec{p}_0)$

$(\vec{c} - \vec{p}_0) \cdot (\vec{p} - \vec{p}_0) = 0$

$\vec{c} \cdot \vec{p} - \vec{c} \cdot \vec{p}_0 - \vec{p}_0 \cdot \vec{p} + \vec{p}_0 \cdot \vec{p}_0 = 0$

(1) $\vec{p}_0 \cdot \vec{p}_0 - \vec{c} \cdot \vec{p}_0 = -\vec{c} \cdot \vec{p} + \vec{p}_0 \cdot \vec{p}$ Line

$(\vec{p}_0 - \vec{c}) \cdot (\vec{p}_0 - \vec{c}) = r^2$

$\vec{p}_0 \cdot \vec{p}_0 - \vec{p}_0 \cdot \vec{c} - \vec{p}_0 \cdot \vec{c} + \vec{c} \cdot \vec{c} = r^2$

(2) $\vec{p}_0 \cdot \vec{p}_0 - \vec{c} \cdot \vec{p}_0 = r^2 + \vec{c} \cdot \vec{p}_0 - \vec{c} \cdot \vec{c}$ circle

(1,2) $\Rightarrow$ 3) $-\vec{c} \cdot \vec{p} + \vec{p}_0 \cdot \vec{p} = r^2 + \vec{c} \cdot \vec{p}_0 - \vec{c} \cdot \vec{c}$

$\vec{c} \cdot \vec{c} - \vec{c} \cdot \vec{p} + \vec{p}_0 \cdot \vec{p} - \vec{p}_0 \cdot \vec{c} = r^2$

$\vec{c}\,(\vec{c} - \vec{p}) + \vec{p}_0\,(\vec{p} - \vec{c}) = r^2$

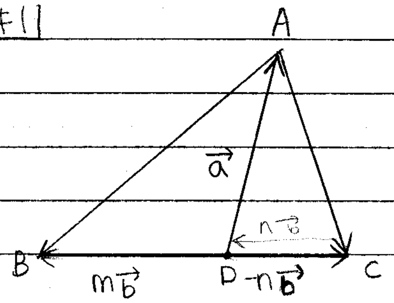
$-\vec{c}\,(\vec{p} - \vec{c}) + \vec{p}_0\,(\vec{p} - \vec{c}) = r^2$

$(\vec{p} - \vec{c}) \cdot (-\vec{c} + \vec{p}_0) = r^2$

$\therefore \quad (\vec{p} - \vec{c}) \cdot (\vec{p}_0 - \vec{c}) = r^2$ EQN line tangent to circle at P_0

$\square$

P68 #11



Prove

$$nAB^2 + mAC^2 = nBD^2 + mCD^2 + (m+n)AD^2$$

Proof

$$\begin{aligned}
LHS &= nAB^2 + mAC^2 \\
&= n(m\vec{b} - \vec{a})^2 + m(-n\vec{b} - \vec{a})^2 \\
&= n(|m\vec{b}|^2 - 2m\vec{a}\cdot\vec{b} + |\vec{a}|^2) + m(|-n\vec{b}|^2 + 2n\vec{a}\cdot\vec{b} + |\vec{a}|^2) \\
&= n|m\vec{b}|^2 - \cancel{2nm\vec{a}\cdot\vec{b}} + n|\vec{a}|^2 + m|-n\vec{b}|^2 + \cancel{2nm\vec{a}\cdot\vec{b}} + m|\vec{a}|^2 \\
&= n|m\vec{b}|^2 + m|-n\vec{b}|^2 + (n+m)|\vec{a}|^2 \\
&= nDB^2 + mDC^2 + (n+m)DA^2 \\
&= nBD^2 + mCD^2 + (n+m)AD^2 = RHS
\end{aligned}$$

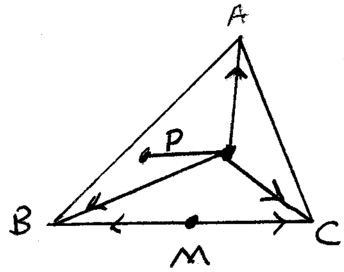
$\square$

This is DaBeen's proof.

P70

[2] Given:

Suppose H orthocenter of ΔABC. LET P be point such that



$$\vec{HP} = \frac{1}{2}(\vec{HA} + \vec{HB} + \vec{HC}).$$

Take H as origin of coordinate system so that position vectors of A, B, C, M, P are $\vec{a}, \vec{b}, \vec{c}, \vec{m}, \vec{p}$.

[2.1] Prove: M midpoint of BC $\Longrightarrow$ $\vec{HA} = 2\vec{MP}$.

Proof, suppose M midpt of BC.

$$\vec{p} = \frac{1}{2}(\vec{a} + \vec{b} + \vec{c})$$

(1) $\quad 2\vec{p} = \vec{a} + \vec{b} + \vec{c}$

$\vec{b} - \vec{m} = -(\vec{c} - \vec{m})$, since M midpt of BC.

$\vec{b} - \vec{m} = \vec{m} - \vec{c}$

(2) $\quad 2\vec{m} = \vec{b} + \vec{c}$

1, 2 $\Longrightarrow$ $2\vec{p} = \vec{a} + 2\vec{m}$

$\vec{a} = 2\vec{p} - 2\vec{m}$

$= 2(\vec{p} - \vec{m})$

so that

$\therefore \quad \vec{HA} = 2\vec{MP}$

$\square$

$\longrightarrow$

[2.2] Prove: P is the circumcenter of ΔABC. That is, P is the intersection of the perpendicular bisectors of the sides of ΔABC.

<u>Proof</u> since H is orthocenter of ΔABC, $\vec{a}, \vec{b}, \vec{c}$ are perpendicular to the sides of ΔABC. It remains to be shown that M is the midpt of BC. Proofs for the two other midpts is identical to that for M of side BC.

$$\vec{a} \cdot (\vec{b} - \vec{m}) = (\vec{a} \cdot \vec{b}) - (\vec{a} \cdot \vec{m})$$
$$\vec{a} \cdot (\vec{c} - \vec{m}) = (\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{m})$$

So,
$$\vec{a} \cdot (\vec{b} - \vec{m}) - (\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\vec{c} - \vec{m}) - (\vec{a} \cdot \vec{c})$$
$$\vec{a} \cdot (\vec{b} - \vec{m}) - \vec{a} \cdot (\vec{c} - \vec{m}) = (\vec{a} \cdot \vec{b}) - (\vec{a} \cdot \vec{c})$$
$$= \vec{a} \cdot (\vec{b} - \vec{c})$$
$$= 0, \quad \text{since } \vec{a} \perp \overrightarrow{CB}$$

Then
$$\vec{a} \cdot (\vec{b} - \vec{m}) = \vec{a} \cdot (\vec{c} - \vec{m})$$
$$\vec{b} - \vec{m} = \vec{c} - \vec{m}$$

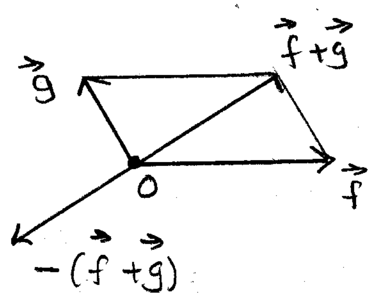
Thus
$$BM = CM$$

Therefore, M midpoint of BC, exactly what we wished to show.

$\square$

P71

[1]

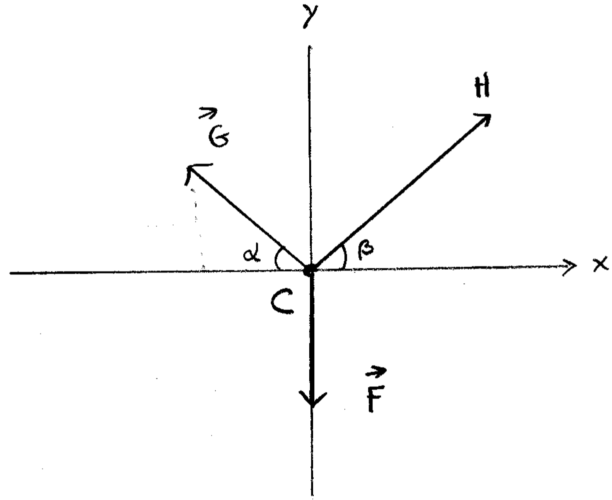


$$\vec{f}+\vec{g}+\vec{h}=0$$

$$\vec{h}=-(\vec{f}+\vec{g})$$

[2]

No movement $\Rightarrow$ Sum of Forces at C = ZERO.



$$G_x = |\vec{G}| \cos\alpha = \tfrac{3}{5} G$$
$$G_y = |\vec{G}| \sin\alpha = \tfrac{4}{5} G$$
$$H_x = |\vec{H}| \cos\beta = \tfrac{4}{5} H$$
$$H_y = |\vec{H}| \sin\beta = \tfrac{3}{5} H$$

$$\vec{F} = -9.8 \tfrac{m}{s^2} (30\,kg)$$
$$= -294\ N$$

$$\cos\alpha = \tfrac{3}{5} \quad \sin\alpha = \tfrac{4}{5}$$
$$\cos\beta = \tfrac{4}{5} \quad \sin\beta = 3/5$$

$$G_x + H_x = 0$$
$$G_y + H_y = 294\ N$$

$$\begin{bmatrix} -\tfrac{3}{5} G + \tfrac{4}{5} H = 0 \\ \tfrac{4}{5} G + \tfrac{3}{5} H = 294 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3G + 4H = 0 \\ 4G + 3H = 1470 \end{bmatrix}$$

$\Rightarrow G = \quad N,\ H = \quad N$

$$G_x = 235\ (3/5) = 141\ N \qquad H_x = 176\ (\tfrac{4}{5}) = 141\ N$$
$$G_y = 235\ (\tfrac{4}{5}) = 188\ N \qquad H_y = 176\ (\tfrac{3}{5}) = 106\ N$$

$\therefore$ $G = \langle -141\ N, 188\ N \rangle$

$H = \langle 141\ N, 106\ N \rangle$

P71, ctd

[3] $\vec{V}_0 = \vec{V}_{river} = \langle 2\frac{m}{s}, 0 \rangle$

$\vec{V} = \vec{V}_{BOAT} = \langle 0, 2\frac{m}{s} \rangle$

$\vec{V}_{actual} = \langle 2\frac{m}{s}, 0 \rangle + \langle 0, 2\frac{m}{s} \rangle$

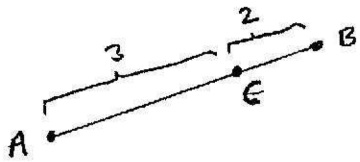
$= \langle 2\frac{m}{s}, 2\frac{m}{s} \rangle$

$\therefore \quad = 2\sqrt{2} \cdot \frac{m}{s}$ in direction NE

≈ 2.8 m/s NE

P72

[1.1]



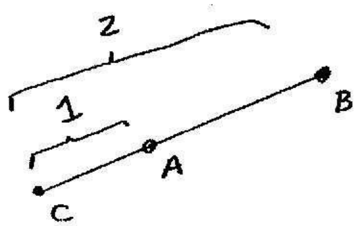
$$\vec{AC} = \frac{3}{5}\vec{AB}$$

$$\vec{c} - \vec{a} = \frac{3}{5}(\vec{b} - \vec{a})$$

$$\vec{c} = \frac{3}{5}\vec{b} - \frac{3}{5}\vec{a} + \vec{a}$$

$$\vec{c} = \frac{2}{5}\vec{a} + \frac{3}{5}\vec{b}$$

[1.2]

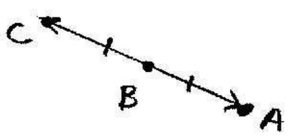


$$\vec{CA} = \vec{AB}$$

$$\vec{a} - \vec{c} = \vec{b} - \vec{a}$$

$$\vec{c} = 2\vec{a} - \vec{b}$$

[1.3]



$$\vec{BC} = -\vec{BA}$$

$$\vec{c} - \vec{b} = \vec{b} - \vec{a}$$

$$\vec{c} = 2\vec{b} - \vec{a}$$

p72, ctd

[2.1]

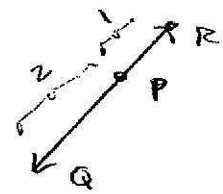
$$\vec{PQ} = \vec{q} - \vec{p} = -6\vec{a} + 6\vec{b} - 2\vec{a} - 2\vec{b} = -8\vec{a} + 4\vec{b}$$

$$\vec{PR} = \vec{r} - \vec{p} = 6\vec{a} - 2\vec{a} - 2\vec{b} = 4\vec{a} - 2\vec{b}$$

[2] Since $\vec{PQ} = -4(2\vec{a} - \vec{b})$

and $\vec{PR} = 2(2\vec{a} - \vec{b})$,

$\vec{PQ} = -2\,\vec{PR}$.



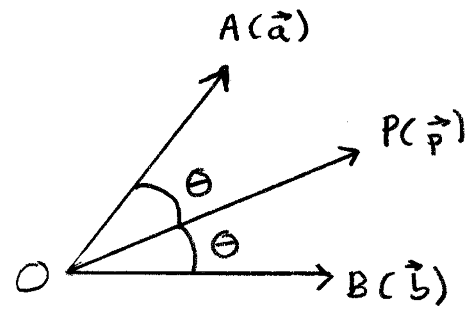
$\therefore$ Points P, Q, R colinear

In fact, P internally divides QR in ratio 2:1

[3.1]

$\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, $\vec{p} = \overrightarrow{OP}$.

$\vec{p}$ bisects angle AOB.



then

$$\cos\theta = \frac{\vec{a}\cdot\vec{p}}{|\vec{a}|\,|\vec{p}|} = \frac{\vec{b}\cdot\vec{p}}{|\vec{b}|\,|\vec{p}|}$$

So $$\frac{\vec{a}\cdot\vec{p}}{|\vec{p}|} = \frac{\vec{b}\cdot\vec{p}}{|\vec{p}|}, \quad \because\ |\vec{a}| = 1 = |\vec{b}|$$

$$\therefore\quad \vec{a}\cdot\vec{p} = \vec{b}\cdot\vec{p}$$

[3.2] as above, then

$$\frac{\vec{a}\cdot\vec{p}}{2} = \frac{\vec{b}\cdot\vec{p}}{3}$$

$$\therefore\quad \vec{a}\cdot\vec{p} = \tfrac{2}{3}(\vec{b}\cdot\vec{p})$$

72, ctd

ANSWER

[4.1] $$\boxed{\begin{array}{ll} l_1: & \vec{p} = \langle 1,1\rangle + s\langle 1,2\rangle \\ l_2: & \vec{p} = \langle 1,5\rangle + t\langle 3,-4\rangle \end{array}}$$

OR

$$\boxed{l_1\left\{\begin{array}{l} x = 1+s \\ y = 1+2s \end{array}\right. \qquad \left.\begin{array}{l} x = 1+3t \\ y = 5-4t \end{array}\right\} l_2}$$

IS OK, too.

* In question 11 ... "taking components of l_1 and l_2 as the parameters s and t". I have no idea what this might mean.

[4.2] $$\begin{bmatrix} 1+s = 1+3t \\ 1+2s = 5-4t \end{bmatrix} \sim \begin{bmatrix} s-3t=0 \\ 2s+4t=4 \end{bmatrix} \sim \begin{bmatrix} 2s-6t=0 \\ 2s+4t=4 \end{bmatrix}$$

$$\Rightarrow 10t = 4 \qquad s - 3\left(\tfrac{2}{5}\right) = 0$$

$$\boxed{t = \tfrac{2}{5}} \qquad \boxed{s = \tfrac{6}{5}}$$

Then, to get coords of pt of intersection

$$x = 1+s = 1+\tfrac{6}{5} = \tfrac{11}{5}$$

$$y = 1+2s = 1+\tfrac{12}{5} = \tfrac{17}{5}$$

$\therefore$ l_1 and l_2 intersect at the Point $\left(\frac{11}{5}, \frac{17}{5}\right)$

[5.1] $$\vec{AC} + \vec{BD} = 2\vec{AD}$$

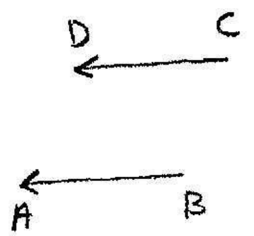
$$\Leftrightarrow \vec{c} - \vec{a} + \vec{d} - \vec{b} = 2(\vec{d} - \vec{a})$$

$$\Leftrightarrow \vec{c} - \vec{a} + \vec{d} - \vec{b} = 2\vec{d} - 2\vec{a}$$

$$\Leftrightarrow \vec{a} - \vec{b} + \vec{c} - \vec{d} = 0$$

$$\Leftrightarrow \vec{a} - \vec{b} = \vec{d} - \vec{c}$$

$$\Leftrightarrow \vec{BA} = \vec{CD}$$



$\therefore$ Since opposite sides are parallel and equal, ABCD is parallelogram

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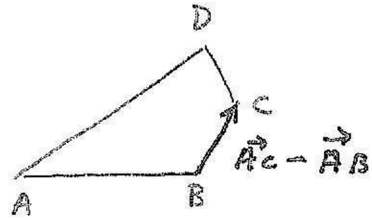
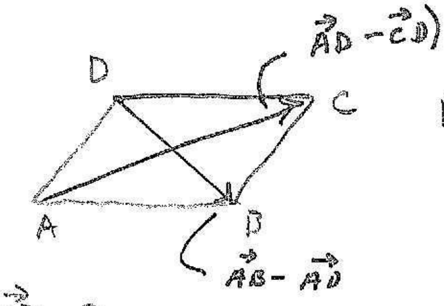
[5.2]



a) $\vec{AC} - \vec{AB} = \vec{BC}$, Since
$\vec{AD} = \vec{BC}$, Figure is parallelogram.



Parallelogram

b) ($\vec{AB} - \vec{AD}$) and ($\vec{AD} - \vec{CD}$) are the diagonals of the parallelogram. Since the dot product is zero, diagonals perpendicular.

$\vec{AD} - \vec{CD}$
$= \vec{AD} + \vec{DC}$ Think this

∴ ABCD is a rhombus.

p73

[1.1]

$$\vec{PO} = \vec{AB} \iff -\langle x, y\rangle = \langle 2, -1\rangle - \langle -3, 4\rangle$$
$$\iff \langle x, y\rangle = -\langle 5, -5\rangle$$
$$= \langle -5, 5\rangle$$

$\therefore$ $P(-5, 5)$

[1.2]

$$\vec{AQ} = \frac{1}{2}\vec{AB} \iff \langle x, y\rangle - \langle -3, 4\rangle = \frac{1}{2}\langle 5, -5\rangle$$
$$\iff \langle x, y\rangle = \langle \frac{5}{2}, -\frac{5}{2}\rangle + \langle -3, 4\rangle$$
$$= \langle -\frac{1}{2}, \frac{3}{2}\rangle$$

$\therefore$ $Q(-\frac{1}{2}, \frac{3}{2})$

[2] Given: $\vec{OA} = 2\vec{a}$, $\vec{OB} = 3\vec{b}$, $\vec{OC} = 6\vec{a} - 6\vec{b}$, $\vec{OD} = 6\vec{b} - 4\vec{a}$

[2.1] Prove A, B, C co-linear.

Proof :

We will show that $\vec{AB} = k\vec{AC}$.

$$\vec{AB} = \vec{OB} - \vec{OA} = 3\vec{b} - 2\vec{a}$$
$$\vec{AC} = \vec{OC} - \vec{OA} = 6\vec{a} - 6\vec{b} - 2\vec{a} = 4\vec{a} - 6\vec{b}$$

Then $-2(-2\vec{a} + 3\vec{b}) = 4\vec{a} - 6\vec{b}$.

Since Both $\vec{AB}$ and $\vec{AC}$ go through Point A and $\vec{AC}$ is a scalar multiple of $\vec{AB}$, A, B, C lie on the same line.

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[2.2] Prove $\vec{AB} \parallel \vec{OD}$

$$\vec{AB} = \vec{OB} - \vec{OA} = 3\vec{b} - 2\vec{a}$$

$$\vec{OD} = 6\vec{b} - 4\vec{a} = 2(3\vec{b} - 2\vec{a}) = 2\vec{AB}.$$

Since $\vec{OD} = 2\vec{AB}$, $\vec{OD} \parallel \vec{AB}$.

[3] Given $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$, $|\vec{a}+\vec{b}| = 1$

[3.1] $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = 1$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}\cdot\vec{b} = 1$$

$$\Rightarrow \vec{a}\cdot\vec{b} = \frac{1 - |\vec{a}|^2 - |\vec{b}|^2}{2}$$

$$= \frac{1-3-4}{2}$$

$$= \frac{-6}{2}$$

$$\therefore \vec{a}\cdot\vec{b} = -3$$

[3.2] $(\vec{a}-\vec{b})\cdot(\vec{a}+2\vec{b})$

$$= |\vec{a}|^2 - 2|\vec{b}|^2 + \vec{a}\cdot\vec{b}$$

$$= 3 - 2(4) - 3$$

JUST showed $\vec{a}\cdot\vec{b} = -3$

$$= -8$$

$$\therefore \vec{a}\cdot\vec{b} = -8$$

[4] Given: $\vec{OP} = \langle 1,1 \rangle$, $\vec{OQ} = \langle 1-\sqrt{3},\ 1+\sqrt{3} \rangle$

[4.1] Get angle made by $\vec{OP}$ and $\vec{OQ}$.

$$\cos\theta = \frac{\vec{OP} \cdot \vec{OQ}}{|\vec{OP}|\,|\vec{OQ}|}$$

$$= \frac{(1-\sqrt{3}) + (1+\sqrt{3})}{\sqrt{1+1}\ \sqrt{(1-\sqrt{3})^2 + (1+\sqrt{3})^2}}$$

$$= \frac{2}{\sqrt{2}\ \sqrt{1+3-2\sqrt{3} + 1+3+2\sqrt{3}}}$$

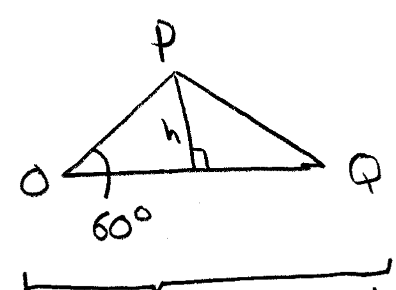
$$= \frac{2}{\sqrt{2}\sqrt{8}} = \frac{2}{4}$$

$$= \frac{1}{2}$$

$$\cos\theta = \frac{1}{2}$$

$$\therefore\ \theta = 60^\circ$$

[4.2] Area of ΔOPQ



You do not need correct shape of ΔOPQ, ONLY the hgt and BASE.

$$A_{\Delta OPQ} = \frac{1}{2} h b$$

$$= \frac{1}{2}\left(|\vec{OP}| \sin 60^\circ\right)\left(|\vec{OQ}|\right)$$

$$= \frac{1}{2}\left[\sqrt{2} \cdot \frac{\sqrt{3}}{2}\right]\sqrt{8}$$

$$= \frac{\sqrt{2}\ \sqrt{3}\ \sqrt{8}}{4}$$

$$\therefore\ A_{\Delta OPQ} = \sqrt{3} \quad \text{SQR units}$$

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[5] $\ell_1 : \sqrt{3}x + y - 1 = 0$

$\ell_2 : x + \sqrt{3}y + 2 = 0$, get angle between lines.

$\vec{n}_1 = \langle \sqrt{3}, 1 \rangle$

$\vec{n}_2 = \langle 1, \sqrt{3} \rangle$

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

$$= \frac{\sqrt{3} + \sqrt{3}}{\sqrt{4}\ \sqrt{4}}$$

$$= \frac{2\sqrt{3}}{4}$$

$$= \frac{\sqrt{3}}{2}$$

$\therefore \quad \theta = 30^\circ$

[6] Given: $|\vec{p}\cdot\vec{q}| \le |\vec{p}||\vec{q}|$. $\vec{p} = \langle a, b\rangle$, $\vec{q} = \langle x, y\rangle$

<u>Prove</u> $(ax+by)^2 \le (a^2+b^2)(x^2+y^2)$

<u>Proof</u>

$$|\vec{p}\cdot\vec{q}| \le |\vec{p}||\vec{q}|$$

$$|\vec{p}\cdot\vec{q}|^2 \le |\vec{p}|^2|\vec{q}|^2$$, since Both sides positive

$$(ax+by)^2 = \left(\sqrt{a^2+b^2}\right)^2\left(\sqrt{x^2+y^2}\right)^2$$

$$\therefore \quad (ax+by)^2 = (a^2+b^2)(x^2+y^2)$$

[7] P has velocity $\vec{v} = \langle 2, 5\rangle$.

At $t=0$, P at $A(-6,-2)$

[7.1] Get $\vec{p}(t)$

$$\vec{p}(t) = \vec{a} + t\vec{v}$$

$$\therefore \quad \vec{p}(t) = \langle -6, -2\rangle + t\langle 2, 5\rangle$$

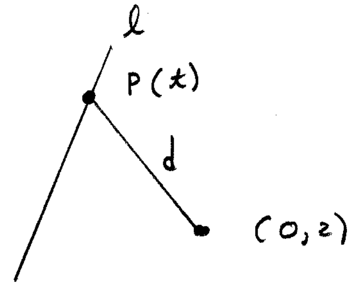
or equivalently

$$\vec{p}(t) = \langle 2t-6, 5t-2\rangle$$

Either answer is O.K.

This solution includes discussion of how to think of it.

[7.2] We are asked to find the value of t for which distance of Line from point $(0,2)$ is minimal.



SOLN

Let P be any point on line.
Then $\vec{P}(t) = \langle 2t-6, 5t-2 \rangle$.
This means that at time t, the Point P has coordinates

$$P(2t-6, 5t-2)$$

The distance formula for two points yields

$$d = \sqrt{[(2t-6)-0]^2 + [(5t-2)-2]^2}$$

so $$d^2 = 4t^2 - 24t + 36 + 25t^2 - 40t + 16$$

$$d^2 = 29t^2 - 64t + 48$$

Since both sides positive, Finding t for which d^2 is minimum is equivalent to findind t for which d is minimum.

By completing the Square, we will find the t for which d^2 is minimum.

$$29t^2 - 64t + 48$$
$$= 29\left[t^2 - \frac{64}{29}t\right] + 48$$
$$= 29\left(t - \frac{64}{2 \cdot 29}\right)^2 + \underbrace{48 - 29\left(\frac{64}{2 \cdot 29}\right)}_{\text{positive}}$$

So d is minimum at $t = \frac{64}{2 \cdot 29} = \frac{32}{29}$ sec

$$\therefore \ t = \frac{32}{29} \text{ sec}$$

→

WITH OUT discussion, soln looks like this

$$d^2 = [(2t-6)-0]^2 + [(5t-2)-2]^2$$

$$= (2t-6)^2 + (5t-4)^2$$

$$= 4t^2 - 24t + 36 + 25t^2 - 40t + 16$$

$$= 29t^2 - 64t + 48$$

$$= 29\left[t^2 - \frac{64}{29}\right] + 48$$

$$= 29\left(t - \frac{64}{2 \cdot 29}\right)^2 + 48 - 29\left(\frac{64}{2 \cdot 29}\right)$$

$\therefore$ d is minimum when $t = \frac{32}{29}$

[1.1]

$\overrightarrow{BA} = \vec{a}, \quad \overrightarrow{BC} = \vec{b}.$

$$\overrightarrow{CE} = \overrightarrow{CB} + \tfrac{1}{3}\overrightarrow{BA}$$
$$= -\overrightarrow{BC} + \tfrac{1}{3}\overrightarrow{BA}$$

① $\boxed{\overrightarrow{CE} = -\vec{b} + \tfrac{1}{3}\vec{a}}$

$$\overrightarrow{CF} = \overrightarrow{CB} + \tfrac{1}{4}\overrightarrow{BD}$$
$$\overrightarrow{BD} = \overrightarrow{CD} - \overrightarrow{CB}$$
$$= \overrightarrow{BA} - \overrightarrow{CB}$$
$$\overrightarrow{CF} = \overrightarrow{CB} + \tfrac{1}{4}(\overrightarrow{BA} - \overrightarrow{CB})$$
$$= \overrightarrow{CB} - \tfrac{1}{4}\overrightarrow{CB} + \tfrac{1}{4}\overrightarrow{BA}$$
$$= \tfrac{3}{4}\overrightarrow{CB} + \tfrac{1}{4}\overrightarrow{BA}$$

② $\boxed{\overrightarrow{CF} = -\tfrac{3}{4}\vec{b} + \tfrac{1}{4}\vec{a}}$

[1.2]

$$\overrightarrow{CE} = -\vec{b} + \tfrac{1}{3}\vec{a} \iff 3\overrightarrow{CE} = -3\vec{b} + \vec{a}$$
$$\overrightarrow{CF} = -\tfrac{3}{4}\vec{b} + \tfrac{1}{4}\vec{a} \iff 4\overrightarrow{CF} = -3\vec{b} + \vec{a}$$

so, $3\overrightarrow{CE} = 4\overrightarrow{CF}$

$$\overrightarrow{CE} = \tfrac{4}{3}\overrightarrow{CF}$$

$\therefore$ F lies on CE

Problem #2 intentionally exclu

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[3]

$$s = |\vec{a}|\,h$$

$$s^2 = |\vec{a}|^2 h^2$$

$$\cos\theta = \frac{h}{|\vec{b}|}$$

$$r^2 = |\vec{b}|^2 \cos\theta$$

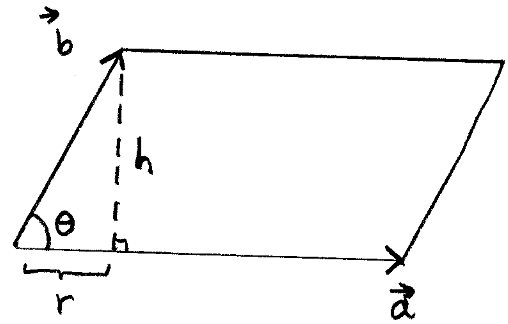
$$h^2 = |\vec{b}|^2 - r^2$$

$$h^2 = |\vec{b}|^2 - |\vec{b}|^2 \cos^2\theta$$

$$s^2 = |\vec{a}|^2 [|\vec{b}|^2 - |\vec{b}|^2 \cos^2\theta$$

$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

$$\therefore\ s^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$$



Area $= s = |\vec{a}|\,h$

Prove $\quad s = |a_1b_2 - a_2b_1|$, $\vec{a} = \langle a_1, a_2\rangle$, $\vec{b} = \langle b_1, b_2\rangle$

<u>Proof</u>

$$s^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$$

$$= (a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2$$

$$= \cancel{a_1^2b_1^2} + a_1^2b_2^2 + a_2^2b_1^2 + \cancel{a_2^2b_2^2} - [\cancel{a_1^2b_1^2} + \cancel{a_2^2b_2^2} + 2a_1b_1a_2b_2]$$

$$= a_1^2b_2^2 + a_2^2b_1^2 - 2a_1b_1a_2b_2$$

$$s^2 = (a_1b_2 - a_2b_1)^2$$

$$\therefore\ s = |a_1b_2 - a_2b_1|$$

[4]

$$\vec{a} \neq 0, \ |\vec{b}| = 2|\vec{a}|, \ \langle \vec{a} + \vec{b} \rangle \perp \langle 5\vec{a} - 2\vec{b} \rangle$$

$$\langle \vec{a} + \vec{b} \rangle \cdot \langle 5\vec{a} - 2\vec{b} \rangle = 0$$

$$\vec{a} \cdot 5\vec{a} - \vec{a} \cdot 2\vec{b} + 5\vec{a} \cdot \vec{b} - 2\vec{b} \cdot \vec{b} = 0$$

$$5|\vec{a}|^2 - 2|\vec{b}|^2 - 2\vec{a} \cdot b + 5\vec{a} \cdot \vec{b} = 0$$

$$5|\vec{a}|^2 - 8|\vec{a}|^2 + 3\vec{a} \cdot \vec{b} = 0$$

$$-3|\vec{a}|^2 + 3\vec{a} \cdot \vec{b} = 0$$

$$3\vec{a} \cdot \vec{b} = 3|\vec{a}|^2$$

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} = 1$$

$$\frac{\vec{a} \cdot \vec{b}}{\frac{1}{4}|\vec{a}||\vec{b}|} = 1$$

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{4}$$

$$\cos\theta = \frac{1}{4}$$

$$\therefore \ \theta = 1.3 \text{ Rad} = 75.5^\circ$$